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Stationary Points for Multifunctions on Two Complete Metric Spaces

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Abstract. In this paper we prove a general fixed point theorem for multifunctions on two complete metric spaces which generalizes the main results from [2] and [5].

1. Introduction

Let (X, d) be a complete metric space and let $B(X)$ be the set of all nonempty subsets of X. As in [1] we define the function $\delta(A, B)$ with A and B in $B(X)$ by $\delta(A, B) = sup\{d(a, b) : a \in A, b \in B\}$.
If A is consists of a single point we write $\delta(A, B) = \delta(a, B)$. If B also consists of single point b then $\delta(a, b) = d(a, b)$. It follows immediately that: $\delta(A, B) = \delta(B, A) \geq 0$ and $\delta(A, B) \leq \delta(A, C) + \delta(C, B)$ for A, B, C in $B(X)$. If $\delta(A, B) = 0$ then $A = B = \{a\}$.
Now if $\{A_n : n = 1, 2, \dots\}$ is a sequence in $B(X)$, we say that it converges to the set A in $B(X)$ if:

(i) each point $a \in A$ is limit of some convergent sequence$\{a_n \in A_n : n = 1, 2, \dots\}$;
(ii) for arbitrary $\epsilon > 0$, there exists an integer N such that $A_n \subset A_\epsilon$ for $n > N$, where A_ϵ is the union of all open spheres with centers in A of radius ϵ.

The set A is said to be limit of the sequence $\{A_n\}$.
The following Lemma was proved in [1].

Lemma 1.1. *If $\{A_n\}$ and $\{B_n\}$ are sequences of bounded subsets of a complete metric space (X, d) which converges to the bounded subsets A and B, respectively, then the sequence $\{\delta(A_n, B_n)\}$ converges to $\delta(A, B)$.*

Let T be a multifunction of X into $B(X)$. z is a stationary point of T if $Tz = \{z\}$.

In 1981, Fisher [2] initiated the study of fixed points on two metric spaces. In 1991, the present author [5] proved other theorems on two metric spaces.

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The following fixed points theorems are proved in [2], resp. [5].

Theorem 1.1 ([2]). *. Let (X,d) and (Y,ρ) be complete metric spaces. If T is a mapping of X into Y and S is a mapping of Y into X satisfying the inequalities*

$$\rho(Tx,TSy) \le c\max\{d(x,Sy),\rho(y,Tx),\rho(y,TSy)\},$$
$$d(Sy,STx) \le c\max\{\rho(y,Tx),d(x,Sy),d(x,STx)\}$$

for all x in X and y in Y, where $0 \le c < 1$, then ST has a unique fixed point z in X and TS has a unique fixed point w in Y. Further, $Tz = w$ and $Sw = z$.

Theorem 1.2 ([5]). *Let (X,d) and (Y,e) be complete metric spaces. If T is a mapping of X into Y and S is a mapping of Y into X satisfying the inequalities*

$$e^2(Tx,TSy) \le c_1 max\{d(x,Sy)e(y,Tx), d(x,Sy)e(y,TSy), e(y,Tx)e(y,TSy)\},$$
$$d^2(Sy,STx) \le c_2 max\{e(y,Tx)d(x,Sy), e(y,Tx)d(x,STx), d(x,Sy)d(x,STx)\}$$

for all x in X and y in Y, where $0 \le c_1, c_2 < 1$, then ST has a unique fixed point z in X and TS has a unique fixed point w in Y. Furthermore, $Tz = w$ and $Sw = z$.

Recently, some fixed points theorems for multifunctions on two complete metric spaces have been proved in [3], [4], [6].

In this paper we prove two generalizations of Theorems 1 and 2 for single valued and set valued mappings satisfying two implicit relations.

2. Implicit relations

Let $\mathcal{F}_4$ be the set of all continuous functions $F : R_+^4 \to R$ such that:

(F_1) : F is nonincreasing in variables t_2, t_3;

(F_2) : there exists $h \in [0,1)$ such that for every $u \ge 0, v \ge 0$ with:
a) $F(u,0,u,v) \le 0$ or b) $F(u,u,0,v) \le 0$
we have $u \le hv$.

Example 2.1. $F(t_1,\dots,t_4) = t_1 - k\max\{t_2,t_3,t_4\}$ where $k \in [0,1)$.

(F_1): Obviously.

(F_2): Let $u > 0$ and $F(u,0,u,v) = u - kmax\{u,v\} \le 0$.

If $u \ge v$ then $u(1-k) \le 0$, a contradiction.
Thus $u < v$ and $u \le hv$. Similarly, $F(u,u,0,v) \le 0$ implies $u \le hv$.
If $u = 0$, then $u \le hv$.

Example 2.2. $F(t_1,\dots,t_4) = t_1^2 - c\max\{t_2t_4, t_2t_3, t_3t_4\}$ where $c \in [0,1)$.

(F_1): Obviously.

(F_2): Let $u > 0$ and $F(u,0,u,v) = u^2 - cuv \le 0$, which implies $u \le hv$, where $h = c \in [0,1)$.

Similarly, $F(u,u,0,v) \le 0$ implies $u \le hv$.
If $u = 0$, then $u \le hv$.

Example 2.3. $F(t_1,\dots,t_4) = t_1^3 - (at_1^2t_2 + bt_3^3 + ct_4^3)$ where $a,b,c > 0$ and $a+b+c<1$.

(F_1): Obviously.

(F_2): $F(u,0,u,v) = u^3 - [bu^3 + cv^3] \le 0$ implies $u \le h_1v$, where $h_1 = (\frac{c}{1-b})^{\frac{1}{3}} < 1$.

Similarly, $F(u,u,0,v) \le 0$ implies $u \le h_2v$, where $h_2 = (\frac{c}{1-a})^{\frac{1}{3}} < 1$. Let $h = \max\{h_1,h_2\}$, then $u \le hv$.

Example 2.4. $F(t_1,\dots,t_4) = t_1 - c\frac{t_2+t_3+t_4}{1+t_4}$ where $0 \le c < \frac{1}{2}$

(F_1): Obviously.

(F_2): $F(u,0,u,v) = u - c\frac{u+v}{1+v}$ implies $u - c(u+v) \le 0$ and $u \le hv$, where $h = \frac{c}{1-c} < 1$. Similarly, $F(u,u,0,v) \le 0$ implies $u \le hv$.

3. Main results

Theorem 3.1. *Let (X,d_1) and (Y,d_2) be two complete metric spaces and let F be a mapping of X into $B(Y)$ and let G be a mapping of Y into $B(X)$ satisfying the inequalities:*

$$\Phi_1(\delta_1(GFx,Gy), d_1(x,Gy), \delta_1(x,GFx), \delta_2(y,Fx)) \le 0 \tag{1}$$

$$\Phi_2(\delta_2(FGy,Fx), d_2(y,Fx), \delta_2(y,FGy), \delta_1(x,Gy)) \le 0 \tag{2}$$

for all x in X and y in Y, where $\Phi_1,\Phi_2 \in \mathcal{F}_4$, then GF has a stationary point z in X and FG has a stationary point w in Y. Furthermore, $Fz = \{w\}$ and $Gw = \{z\}$.

Proof. Let x_1 be an arbitrary point in X. Define sequences $\{x_n\}$ and $\{y_n\}$ in X and Y, respectively, as follows: choose a point y_1 in Fx_1 and a point x_2 in Gy_1. In general, having chosen x_n in X and y_n in Y, we choose x_{n+1} in Gy_n and then y_{n+1} in Fx_{n+1} for $n = 1,2,\dots$.
Then, by (1), we have successively

$$\Phi_1(\delta_1(GFx_{n+1},Gy_n), d_1(x_{n+1},Gy_n), \delta_1(x_{n+1},GFx_{n+1}), \delta_2(y_n,Fx_{n+1})) \le 0$$

$$\Phi_1(\delta_1(GFx_{n+1},Gy_n), 0, \delta_1(Gy_n,GFx_{n+1}), \delta_2(y_n,Fx_{n+1})) \le 0$$

which implies

$$\delta_1(GFx_{n+1},Gy_n) \le h\delta_2(y_n,Fx_{n+1}). \tag{3}$$

By (2) we have successively

$$\Phi_2(\delta_2(FGy_n,Fx_n), d_2(y_n,Fx_n), \delta_2(y_n,FGy_n), \delta_1(x_n,Gy_n)) \le 0$$

$$\Phi_2(\delta_2(FGy_n,Fx_n), 0, \delta_2(Fx_n,FGy_n), \delta_1(x_n,Gy_n)) \le 0$$

which implies

$$\delta_2(FGy_n,Fx_n) \le h_2\delta_1(x_n,Gy_n). \tag{4}$$

Thus, it follows from (3) and (4) that

$$d_1(x_{n+1}, x_{n+2}) \le \delta_1(Gy_n, GFx_{n+1}) \le h_1\delta_2(y_n, Fx_{n+1}) \le h_1\delta_2(Fx_n, GFy_n) \le \\ \le h_1h_2\delta_1(x_n, Gy_n) \le \dots (h_1h_2)^n\delta_1(x_1, GFx_1).$$

Similarly, we can prove that

$$d_2(y_{n+1}, y_n) \le (h_1h_2)^n\delta_2(y_1, FGy_1).$$

Now, it follows that for $n = 1, 2, \dots$ and $r \in N^*$

$$\begin{aligned} d_1(x_{n+1}, x_{n+r+1}) &\le \delta_1(Gy_n, GFx_{n+r}) \le \\ &\le \delta_1(Gy_n, Gy_{n+1}) + \delta_1(Gy_{n+1}, y_{n+2}) + \dots + \\ &\quad + \delta_1(Gy_{n+r-1}, GFx_{n+r}) \le \\ &\le \delta_1(Gy_n, GFx_{n+1}) + \delta_1(Gy_{n+1}, GFx_{n+2}) + \dots + \\ &\quad + \delta_1(Gy_{n+r-1}, GFx_{n+r}) \le \\ &\le \{(h_1h_2)^n + (h_1h_2)^{n+1} + \dots + (h_1h_2)^{n+r-1}\}\delta_1(x_1, GFx_1) < \epsilon \end{aligned}$$

for n greater than some N since $h_1h_2 < 1$.

Therefore, the sequence $\{x_n\}$ is a Cauchy sequence in the complete metric space X and so it has a limit z in X.

Similarly, the sequence $\{y_n\}$ is a Cauchy sequence in the complete metric space Y and so it has a limit w in Y.

Further

$$\begin{aligned} \delta_1(z, GFx_n) &\le d_1(z, x_{m+1}) + \delta_1(x_{m+1}, GFx_n) \le \\ &\le d_1(z, x_{m+1}) + \delta_1(Gy_m, GFx_n) \\ &\le d_1(z, x_{m+1}) + \epsilon \text{ for } m, n > N. \end{aligned}$$

Letting m tend to infinity it follows that

$$\delta_1(z, GFx_n) < \epsilon$$

for $n > N$ and

(5) $$\lim GFx_n = z = \lim Gy_n.$$

Similarly,

(6) $$\lim FGy_n = w = \lim Fx_n.$$

Using inequality (2) and (F_1) we have

$$\Phi_2(\delta_2(FGy_n, Fz), \delta_2(y_n, Fz), \delta_2(y_n, FGy_n), \delta_1(z, Gy_n)) \le 0.$$

Letting n tend to infinity we obtain successively

$$\Phi_2(\delta_2(w, Fz), \delta_2(w, Fz), \delta_2(w, w), \delta_1(z, z)) \le 0$$

$$\Phi_2(\delta_2(w, Fz), \delta_2(w, Fz), 0, 0) \le 0$$

which implies $\delta_2(w, Fz) = 0$. Thus

(7) $$Fz = \{w\}.$$

Similarly, we can prove that

$$Gw = \{z\}. \tag{8}$$

From (7) and (8), it follows that

$$GFz = Gw = \{z\} \quad \text{and} \quad FGw = Fz = \{w\}.$$

Thus z is a stationary point of GF and w is a stationary point of FG. This completes the proof of Theorem 3. □

Theorem 3.2. *Let (X, d_1) and (Y, d_2) be two complete metric spaces and let f be a single valued mapping of X into Y and g a single valued mapping of Y into X satisfying the inequalities*

$$\Phi_1(d_1(gfx, gy), d_1(x, gy), d_1(x, gfx), d_2(y, fx)) \leq 0 \tag{1'}$$

$$\Phi_2(d_2(fgy, fx), d_2(y, fx), d_2(y, fgy), d_1(x, gy)) \leq 0 \tag{2'}$$

for all x in X and y in Y, where $\Phi_1, \Phi_2 \in \mathcal{F}_4$.

Then gf has an unique fixed point z in X and fg has an unique fixed point w in Y. Further, $fz = w$ and $gw = z$.

Proof. The existence of z and w follows from Theorem 3. Now suppose that gf has a second fixed point z'.

Then by $(1')$ we have successively

$$\Phi_1(d_1(gfz, gfz'), d_1(z, gfz'), d_1(z, gfz), d_2(fz', fz)) \leq 0$$

$$\Phi_1(d_1(z, z'), d(z, z'), 0, d(fz, fz')) \leq 0$$

which implies

$$d(z, z') \leq h_1 d(fz, fz'). \tag{9}$$

Similarly, by $(2')$ we have successively

$$\Phi_2(d_2(fgfz, fz'), d_2(fz, fz'), d_2(fz, fgfz), d_1(z', gfz)) \leq 0$$

$$\Phi_2(d_2(fz, fz'), d(fz, fz'), 0, d(z, z')) \leq 0$$

which implies

$$d_2(fz, fz') \leq h_2 d(z, z'). \tag{10}$$

By (9) and (10) we have

$$d_1(z, z_1) \leq h_1 d_2(fz, fz') \leq (h_1 h_2) d_1(z, z').$$

Since $h_1 h_2 < 1$ it follows that $z = z'$.

Similarly fg has a unique fixed point. □

Corollary 3.1. *Theorem 1.1.*

Proof. The proof follows from Theorem 3.2 and Example 1. □

Corollary 3.2. *Theorem 1.2.*

Proof. The proof follows from Theorem 3.2 and Example 2. □

REFERENCES

[1] Fisher, B., *Common fixed points of mappings and set valued mappings*, Rostok Math.Kolloq. **8**(1981), 68–77.

[2] Fisher, B., *Fixed points on two metric spaces*, Glasnik Mat. **16**(**36**),(1981), 333–337.

[3] Fisher, B. and Türköglu,D., *Related fixed point for set valued mappings on two metric spaces*, Internat. J.Math.Math.Sci. **23**(2000), 205–210.

[4] Liu Z., Liu Q., *Stationary points for set-valued mappings on two metric spaces*, Rostok Math. Kolloq. **55**(2001), 23–29.

[5] Popa V., *Fixed points on two complete metric spaces*, Zb.Rad.Prirod.Mat.Fiz. Ser.Mat.Univ. Novom Sadu, **21**(1991),83–93.

[6] Popa V., *Two general fixed point theorem for mappings on two metric spaces*, Anal.Univ.Galaţi, fasc.II. Mat.Fiz.Mec.Teor.Suppl. vol.18(**23**)(2001),19–24.

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